

QUESTION 1

Student Number:	e-mail:	Group:	List No.:	Grade
Name:	Surname:	Sign.:		

[12p] a) $\lim_{x \rightarrow 0} \frac{e^x - 1 + \int_0^x e^{t \tan^{-1} t} dt}{2(1 - \cos x)} = ?$

[13p] b) Discuss the convergence of $\int_1^\infty 2e^{-x^2} dx$. Show all work.

a) $\lim_{x \rightarrow 0} \frac{e^x - 1 + \int_0^x e^{t \tan^{-1} t} dt}{2(1 - \cos x)} : \left(\frac{0}{0}\right)$

$\lim_{x \rightarrow 0} \frac{e^x + e^{t \tan^{-1} t} x}{2 \sin x} : \infty$

b) $I = 2 \int_1^\infty e^{-x^2} dx$

$1 \leq x < \infty, \quad x^2 \geq x, \quad -x^2 \leq -x$

$e^{-x^2} \leq e^{-x}, \quad g(x) = e^{-x}, \quad f(x) = e^{-x^2}$

Let $\int_1^\infty g(x) dx = \int_1^\infty e^{-x} dx$

$\int_1^\infty g(x) dx = \lim_{b \rightarrow \infty} \int_1^b e^{-x} dx = \lim_{b \rightarrow \infty} \left[-e^{-x} \right]_1^b$

$= \lim_{b \rightarrow \infty} -e^{-b} + \frac{1}{e} = -\frac{1}{e^\infty} + \frac{1}{e} = \frac{1}{e}$

$\int_1^\infty g(x) dx$ converges \Rightarrow

$\int_1^\infty f(x) dx$ converges

$\Rightarrow \int_1^\infty 2e^{-x^2} dx$ converges

QUESTION 2

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Evaluate the following integrals.

[12p] a) $\int_2^{\infty} \frac{(x+3)}{(x-1)(x^2+1)} dx$

[13p] b) $\int \frac{\tan^{-1}(2x)}{4x^2} dx$

a) $I = \int_2^{\infty} \frac{(x+3)}{(x-1)(x^2+1)} dx$

$$\frac{x+3}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}, \quad A=2, B=-2, C=-1$$

$$\frac{x+3}{(x-1)(x^2+1)} = \frac{2}{x-1} - \frac{2x+1}{x^2+1}$$

$$\int_2^{\infty} \frac{(x+3)}{(x-1)(x^2+1)} dx = 2 \int_2^{\infty} \frac{dx}{x-1} - \int_2^{\infty} \frac{(2x+1)}{(x^2+1)} dx$$

$$= \lim_{b \rightarrow \infty} \left(2 \int_2^b \frac{dx}{x-1} - \int_2^b \frac{2x dx}{x^2+1} - \int_2^b \frac{dx}{x^2+1} \right)$$

$$= \lim_{b \rightarrow \infty} \left(2 \ln|x-1| \Big|_2^b - \ln|x^2+1| \Big|_2^b - \tan^{-1} x \Big|_2^b \right)$$

$$= \lim_{b \rightarrow \infty} \left(\frac{\ln(b-1)^2}{(b^2+1)} - 2 \ln 1 + \ln 5 - \tan^{-1} b + \tan^{-1} 2 \right)$$

$$= \lim_{b \rightarrow \infty} \left[\ln \left(\frac{(b-1)^2}{b^2+1} \right) - \tan^{-1}(b) \right] + \ln 5 + \tan^{-1} 2$$

$$= \ln \left(\lim_{b \rightarrow \infty} \frac{b^2-2b+1}{b^2+1} \right) - \tan^{-1}(\infty) + \ln 5 + \tan^{-1} 2$$

$$= \ln 1 - \frac{\pi}{2} + \ln 5 + \tan^{-1} 2 = \boxed{\ln 5 + \tan^{-1} 2 - \frac{\pi}{2}}$$

$$b) I = \int \frac{\arctan^{-1}(2x)}{4x^2} dx \quad \left[\begin{array}{l} \arctan^{-1}(2x) = u \\ \frac{dx}{4x^2} = du, \quad u = -\frac{1}{4x} \end{array} \right]$$

$$I = -\frac{\arctan^{-1}(2x)}{4x} + \frac{2}{4} \int \frac{1}{x} \cdot \frac{dx}{(1+4x^2)}$$

$$\frac{1}{x(1+4x^2)} = \frac{A}{x} + \frac{Bx+C}{1+4x^2}, \quad A=1, \quad B=-4$$

$$\int \frac{dx}{x(1+4x^2)} = \int \frac{dx}{x} - 4 \int \frac{x dx}{1+4x^2} \quad \left[\begin{array}{l} C=0 \\ 1+4x^2 = m \\ x dx = \frac{dm}{8} \end{array} \right]$$

$$= \ln|x| - \frac{4}{8} \cdot \ln|1+4x^2| + C_1$$

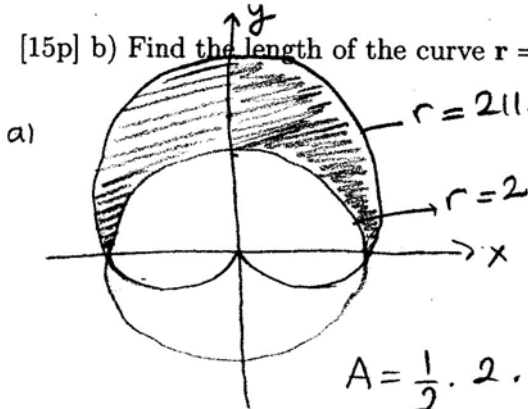
$$I = -\frac{\arctan^{-1} 2x}{4x} + \frac{1}{2} \cdot \ln|x| - \frac{1}{4} \ln(1+4x^2) + C$$

QUESTION 3

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[15p] a) Calculate the area that lies inside the cardioid $r = 2(1 + \sin\theta)$ and outside the circle $r = 2$.

[15p] b) Find the length of the curve $r = \cos^3\left(\frac{\theta}{3}\right)$, $0 \leq \theta \leq \frac{\pi}{4}$.



$$A = \frac{1}{2} \int_{\theta=\alpha}^{\theta=\beta} r^2 d\theta$$

$$2(1 + \sin\theta) = 2, \quad 1 + \sin\theta = 1 \\ \sin\theta = 0 \\ \theta_1 = 0, \quad \theta_2 = \pi$$

$$A = \frac{1}{2} \cdot 2 \cdot \int_{\theta=0}^{\pi/2} 2^2 (1 + \sin\theta)^2 d\theta - \frac{\pi \cdot 2^2}{2}$$

or

$$A = \frac{1}{2} \cdot 2 \cdot \int_{\theta=0}^{\pi/2} [2^2 (1 + \sin\theta)^2 - 2^2] d\theta$$

$$A = \int_0^{\pi/2} 4 [1 + 2\sin\theta + \sin^2\theta - 1] d\theta = 4 \int_0^{\pi/2} 2\sin\theta d\theta + 4 \int_0^{\pi/2} \sin^2\theta d\theta$$

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$A = 8(-\cos\theta) \Big|_0^{\pi/2} + \frac{4}{2} \left(\theta - \frac{\sin 2\theta}{2} \right) \Big|_0^{\pi/2}$$

$$A = 8(-\cos \frac{\pi}{2} + \cos 0) + 2 \left(\frac{\pi}{2} - \frac{\sin \pi}{2} - 0 + \frac{\sin 0}{2} \right) = \boxed{8 + \pi}$$

b)

$$s = \int_{\theta=\alpha}^{\theta=\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

$$\frac{dr}{d\theta} = 3\cos^2\frac{\theta}{3} \cdot \frac{1}{3} \cdot (-\sin\frac{\theta}{3}) = -\cos^2\frac{\theta}{3} \sin\frac{\theta}{3}$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \cos^6\left(\frac{\theta}{3}\right) + \cos^4\frac{\theta}{3} \sin^2\frac{\theta}{3} = \cos^4\frac{\theta}{3} \left(\cos^2\frac{\theta}{3} + \sin^2\frac{\theta}{3} \right)$$

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{\cos^4\frac{\theta}{3}} = |\cos^2\frac{\theta}{3}| = \cos^2\frac{\theta}{3}$$

$$s = \int_{\theta=0}^{\pi/4} \cos^2\frac{\theta}{3} d\theta = \frac{1}{2} \int_{\theta=0}^{\pi/4} (1 + \cos \frac{2\theta}{3}) d\theta$$

$$s = \frac{1}{2} \cdot \left(\theta + \frac{3}{2} \sin \frac{2\theta}{3} \right) \Big|_0^{\pi/4} = \frac{1}{2} \left(\frac{\pi}{4} + \frac{3}{2} \cdot \frac{1}{2} \right) = \boxed{\frac{\pi+3}{8}}$$

QUESTION 4

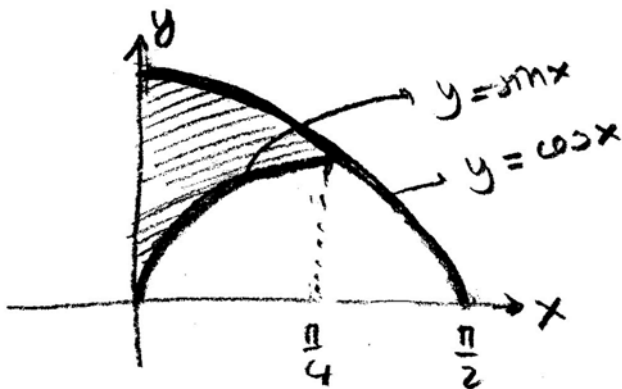
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Consider the region in the first quadrant, bounded by the curves $y = \cos x$, $y = \sin x$ and the y-axis.

Find the volume of the solid generated by revolving this region;

[10p] a) about the y-axis, by using the shell method.

[10p] b) about the x-axis, by using the disk method.



$$a) V_y = 2\pi \int_{x_1}^{x_2} xy \, dx$$

$$V_y = 2\pi \int_0^{\pi/4} x(\cos x - \sin x) \, dx$$

$$\left[\begin{array}{l} x=u, \, dx=du \\ (\cos x - \sin x) \, dx = du \\ (\sin x + \cos x) = u \end{array} \right]$$

$$V_y = 2\pi \left(x(\cos x + \sin x) \Big|_0^{\pi/4} - \int_0^{\pi/4} (\sin x + \cos x) \, dx \right)$$

$$V_y = 2\pi \left(\frac{\pi}{4} \cdot 2 \cdot \frac{\sqrt{2}}{2} + (\cos x - \sin x) \Big|_0^{\pi/4} \right)$$

$$V_y = 2\pi \left(\frac{\sqrt{2}\pi}{4} + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} - 1 \right) = 2\pi \left(\frac{\sqrt{2}}{4}\pi - 1 \right)$$

$$b) V_x = \pi \int_0^{\pi/4} (\cos^2 x - \sin^2 x) \, dx$$

$$V_x = \pi \int_{x_1}^{x_2} [(R(x))^2 - (r(x))^2] \, dx$$

$$V_x = \pi \int_0^{\pi/4} \cos 2x \, dx$$

$$V_x = \pi \frac{\sin 2x}{2} \Big|_0^{\pi/4} = \boxed{\frac{\pi}{2}}$$