MATHEMATICS 101E FINAL 2-January-2013

**QUESTION 1** The blanks below will be filled by students. (Except the score)

| Surname: | Name: KEY | Group Number: | List Number: | **Score** |
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**For the solution of this question please use only the front face and if necessary the back face of this page.**

[10 pts] a) Evaluate the integral $\displaystyle\int \frac{\mathrm{dx}}{\mathrm{x}^2\sqrt{1-\mathrm{x}^2}}$ using trigonometric substitution.

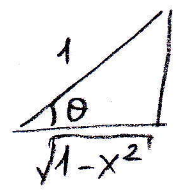
[10 pts] b) Evaluate the integral $\displaystyle\int \frac{\mathrm{x}}{(\mathrm{x}+2)^2(\mathrm{x}+1)}\,\mathrm{dx}$ .

[5 pts] c) Find $\dfrac{\mathrm{dy}}{\mathrm{dx}}$ if $\mathrm{y} = \mathrm{x}^{\mathrm{x}}$.

a)
$$\int \frac{dx}{x^2\sqrt{1-x^2}} = \int \frac{\cos\theta}{\sin^2\theta\sqrt{1-\sin^2\theta}}\,d\theta$$

$x = \sin\theta$

$dx = \cos\theta\, d\theta$




$\cot an\theta = \dfrac{\sqrt{1-x^2}}{x}$

$$= \int \frac{\cancel{\cos\theta}}{\sin^2\theta\ \cancel{\cos\theta}}\,d\theta$$

$$= \int \frac{d\theta}{\sin^2\theta} = \int \operatorname{cosec}^2\theta\, d\theta$$

$$= -\cot an\,\theta + C = -\frac{\sqrt{1-x^2}}{x} + C$$

b)
$$\frac{x}{(x+2)^2(x+1)} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2} + \frac{C}{x+1}$$

$$x = A(x+2)(x+1) + B(x+1) + C(x+2)^2$$

$$\left.\begin{array}{lll} x=-2 & \Rightarrow & -2 = -B \\ x=-1 & \Rightarrow & -1 = C \\ x = 0 & \Rightarrow & 0 = 2A + B + 4C \end{array}\right\} \quad \begin{array}{l} A = 1 \\ B = 2 \\ C = -1 \end{array}$$

$$\int \frac{x}{(x+2)^2(x+1)}\,dx = \int \frac{dx}{x+2} + \int \frac{2}{(x+2)^2}\,dx - \int \frac{dx}{x+1}$$

$$= \ln|x+2| - \frac{2}{(x+2)} - \ln|x+1| + C$$

c) $y = x^x$

$\ln y = \ln(x^x)$

$\ln y = x\ln x$

Taking the derivative of both sides of equatio we have

$$\frac{y'}{y} = 1.\ln x + x.\frac{1}{x} \Rightarrow y' = x^x(1+\ln x)$$

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**QUESTION 2** The blanks below will be filled by students. (Except the score)

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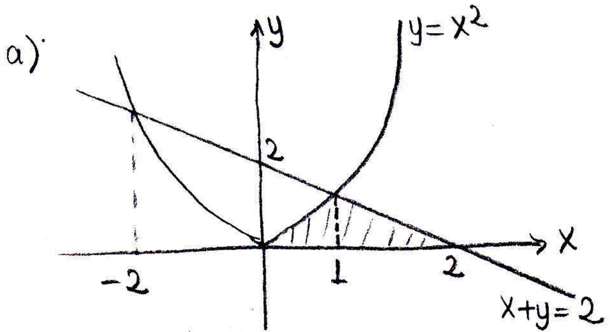
[10 pts] a) Find the area of the region $\mathbf{R}$ enclosed by the curve $\mathbf{y = x^2}$ and the line $\mathbf{x + y = 2}$ and the $\mathbf{x}$−axis by using definite integrals.

[15 pts] b) Find the area of the region $\mathbf{R}$ inside the circle $\mathbf{r = 3\sin\theta}$ and outside the cardioid $\mathbf{r = 1 + \sin\theta}$.

a)

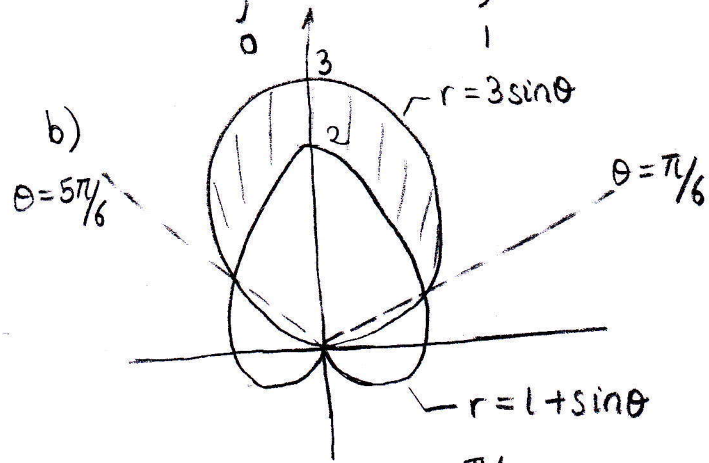



$x^2 = 2 - x \Rightarrow x^2 + x - 2 = 0$

$x = -2 \quad x = 1$ intersection points.

$$\text{Area} = \int_0^1 x^2\,dx + \int_1^2 (2-x)\,dx = \frac{x^3}{3}\Big|_0^1 + \left(2x - \frac{x^2}{2}\right)\Big|_1^2 = \frac{1}{3} + \left(2 - \frac{3}{2}\right) = \frac{5}{6}$$

b)




$1 + \sin\theta = 3\sin\theta$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$$\text{Area} = 2\left[\frac{1}{2}\int_{\pi/6}^{\pi/2} (3\sin\theta)^2\,d\theta - \frac{1}{2}\int_{\pi/6}^{\pi/2} (1+\sin\theta)^2\,d\theta\right]$$

$$= \int_{\pi/6}^{\pi/2} (8\sin^2\theta - 1 - 2\sin\theta)\,d\theta$$

$$= 8\int_{\pi/6}^{\pi/2} \frac{1-\cos(2\theta)}{2}\,d\theta + (-\theta + 2\cos\theta)\Big|_{\pi/6}^{\pi/2}$$

$$= 4\left[\theta - \frac{\sin(2\theta)}{2}\right]\Big|_{\pi/6}^{\pi/2} + \left(-\frac{\pi}{2} + \frac{\pi}{6} - \sqrt{3}\right)$$

$$= 4\left(\frac{\pi}{2} - \frac{\pi}{6} + \frac{\sqrt{3}}{4}\right) + \left(-\frac{\pi}{3} - \sqrt{3}\right) = \pi$$

**QUESTION 3** The blanks below will be filled by students. (Except the score)

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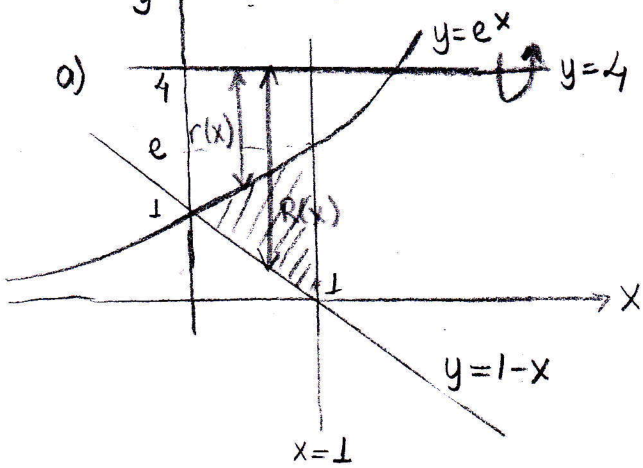
**R** is the region enclosed by the curve $\mathbf{y = e^x}$ and the lines $\mathbf{y = 1 - x}$ and $\mathbf{x = 1}$.

[10 pts] **i)** Write the definite integral that calculates the volume of the solid generated by revolving the region **R** about the line $\mathbf{y = 4}$ by using Disk/Washer method.
(Do not evaluate the definite integral.)

[15 pts] **ii)** Find the the volume of the solid generated by revolving the region **R** about the y-axis by using Shell method.

a)



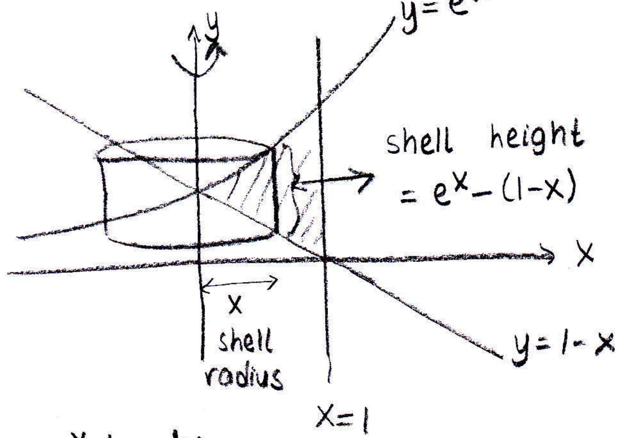

a) $R(x) = 4 - (1-x) = 3 + x$

$r(x) = 4 - e^x$

$$V_{y=4} = \pi \int \{[R(x)]^2 - [r(x)]^2\} dx$$

$$= \pi \int_0^1 \left[(4-(1-x))^2 - (4-e^x)^2\right] dx$$

b)




$$V_y = 2\pi \int \left(\begin{matrix}\text{shell}\\ \text{radius}\end{matrix}\right)\left(\begin{matrix}\text{shell}\\ \text{height}\end{matrix}\right) dx$$

$$= 2\pi \int_0^1 x\,[e^x - (1-x)]\, dx$$

$$= 2\pi \int_0^1 x\,(e^x + x - 1)\, dx$$

$x = u \quad e^x dx = dv$

$dx = du \quad v = e^x$

$$= 2\pi \left[\int_0^1 x e^x dx + \int_0^1 (x^2 - x) dx\right]$$

$$= 2\pi \left[ x e^x \Big|_0^1 - \int_0^1 e^x dx + \left(\frac{x^3}{3} - \frac{x^2}{2}\right)\Big|_0^1 \right] = 2\pi \cdot \frac{5}{6} = \frac{5\pi}{3}$$

**QUESTION 4** The blanks below will be filled by students. (Except the score)

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**For the solution of this question please use only the front face and if necessary the back face of this page.**

[8 pts] a) $\lim\limits_{x\to 0} \dfrac{\int_0^{\sinh x} e^{t^2}\,dt}{2^x - e^{\sin x}} = ?$

[7+10 pts] b) Investigate the convergence or divergence of the following improper integrals.

i) $\int_1^{\infty} \frac{1+\sin^2 x}{\sqrt{x}} dx$ ii) $\int_1^2 \frac{dx}{\sqrt{4-x^2}}$

a) $\lim\limits_{x\to 0} \dfrac{\int_0^{\sinh x} e^{t^2}\,dt}{2^x - e^{\sin x}} \overset{\left(\frac{0}{0}\right)}{=} \lim\limits_{x\to 0} \dfrac{e^{(\sinh x)^2}\cosh x}{2^x.\ln 2 - e^{\sin x}\cos x} = \dfrac{1}{\ln 2 - 1}$.

b) i) $0 < \frac{1}{\sqrt{x}} \leq \frac{1+\sin^2 x}{\sqrt{x}}$ $x \in [1,\infty)$

$\int_1^{\infty} \frac{dx}{\sqrt{x}}$ is divergent since $p = \frac{1}{2} \leq 1$. By the Direct Comparison test, $\int_1^{\infty} \frac{1+\sin^2 x}{\sqrt{x}} dx$ is also divergent.

ii) $\int_1^2 \frac{dx}{\sqrt{4-x^2}} = \lim\limits_{b\to 2^-} \int_1^b \frac{dx}{\sqrt{4-x^2}}$

$= \lim_{b\to 2^-} \sin^{-1}\left(\frac{x}{2}\right)\Big|_1^b$

$= \lim\limits_{b\to 2^-} \left[\sin^{-1}\left(\frac{b}{2}\right) - \sin^{-1}\left(\frac{1}{2}\right)\right]$

$= \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$

Since the limit is finite, $\int_1^2 \frac{dx}{\sqrt{4-x^2}}$ is convergent.