MATHEMATICS 101E RESIT EXAM 16 JANUARY 2013

QUESTION 1 The blanks below will be filled by students. (Except the score)

Surname:	Name: KEY	Group Number:	List Number:	Score
Signature:	Electronic Post(e-mail) address:	Student Number:		

For the solution of this question please use only the front face and if necessary the back face of this page.

Evaluate the following integrals.

[10 pts] **a)** $\displaystyle\int \frac{2y^4}{y^3-y^2+y-1}\,dy$ [8 pts] **b)** $\displaystyle\int_0^1 x\cosh(2x)\,dx$ [12 pts] **c)** $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx$

a). $$\frac{2y^4}{y^3-y^2+y-1} = 2(y+1) + \frac{2}{y^3-y^2+y-1}$$

$$\frac{2}{y^3-y^2+y-1} = \frac{2}{(y-1)(y^2+1)} = \frac{A}{y-1} + \frac{By+C}{y^2+1}$$

$$2 = A(y^2+1) + (By+C)(y-1)$$

$y=1 \Rightarrow 2 = 2A$

$y=0 \Rightarrow 2 = A - C$

$y=-1 \Rightarrow 2 = 2A+2B-2C$

$A=1$, $C=-1$, $B=-1$

$$\int \frac{2y^4}{y^3-y^2+y-1}dy = \int 2(y+1)dy + \int \frac{dy}{y-1} - \int \frac{y}{y^2+1}dy - \int \frac{dy}{y^2+1}$$

$$= y^2+2y+\ln|y-1| - \frac{1}{2}\ln|y^2+1| - \tan^{-1}(y) + C$$

b) $$\int_0^1 x\cosh(2x)\,dx = x.\frac{\sinh(2x)}{2}\Big|_0^1 - \int_0^1 \frac{\sinh(2x)}{2}dx$$

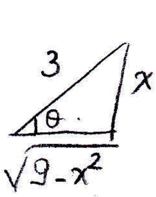
$x=u$, $dx=du$, $\cosh(2x)dx=dv$, $v=\frac{\sinh(2x)}{2}$

$$= \frac{\sinh(2)}{2} - \frac{\cosh(2x)}{4}\Big|_0^1 = \frac{\sinh(2)}{2} - \frac{\cosh(2)}{4} + \frac{1}{4}$$

c) $$\int \frac{\sqrt{9-x^2}}{x^2}dx = \int \frac{3(\cos\theta)}{9\sin^2\theta}.3\cos\theta\, d\theta = \int \frac{\cos^2\theta}{\sin^2\theta}d\theta$$

$x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$



$\cot\theta = \frac{\sqrt{9-x^2}}{x}$

$$= \int (\cot^2\theta + 1 - 1)\, d\theta$$

$$= \int \csc^2\theta - 1\, d\theta$$

$$= -\cot\theta - \theta + C$$

$$= -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$$

QUESTION 2 The blanks below will be filled by students. (Except the score)

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[10 pts] **a)** Evaluate the following limit

$$\lim_{x\to 0} \frac{\int_0^x \sin^{-1} t\, dt}{ln^2(x+1)}$$

[10 pts] **b)** If $f(x) = f(a-x)$, then prove that

$$\int_0^a f(x)\, dx = 2\int_0^{a/2} f(x)\, dx$$

a) $$\lim_{x\to 0} \frac{\int_0^x \sin^{-1} t\, dt}{\ln^2(x+1)} \left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{\sin^{-1} x}{2\ln(x+1)\cdot\frac{1}{x+1}}$$

$$= \lim_{x\to 0} \frac{(x+1)\sin^{-1} x}{2\ln(x+1)} \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{\sin^{-1} x + (x+1)\cdot\frac{1}{\sqrt{1-x^2}}}{\frac{2}{x+1}} = \frac{1}{2}$$

b) $$\int_0^a f(x)dx = \int_0^{a/2} f(x)dx + \int_{a/2}^{a} f(x)dx \qquad \begin{aligned} x &= a-t \\ dx &= -dt \end{aligned}$$

$$= \int_0^{a/2} f(x)dx + \int_{a/2}^{0} f(a-t)(-dt)$$

$$= \int_0^{a/2} f(x)dx + \int_0^{a/2} f(a-t)\,dt = \int_0^{a/2} f(x)dx + \int_0^{a/2} f(t)\,dt = 2\int_0^{a/2} f(x)dx$$

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QUESTION 3 The blanks below will be filled by students. (Except the score)

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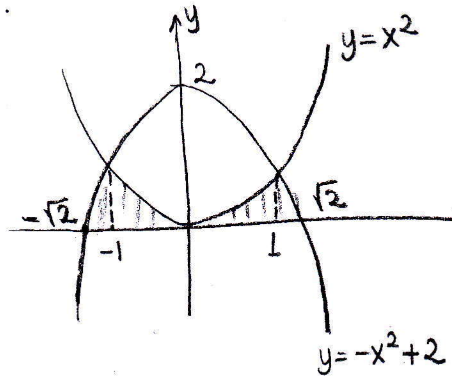
[15 pts] **a)** Find the area of the region R enclosed by the curves $y = x^2$, $y = -x^2 + 2$ and the x−axis.

[15 pts] **b)** Find the volume of the solid generated by revolving the region enclosed by the line $y = x$, the curve $y = \sqrt{1 - x^2}$ and the y−axis about the line $x = 1$ using Shell Method.

a)



Intersection points: $x^2 = -x^2 + 2$

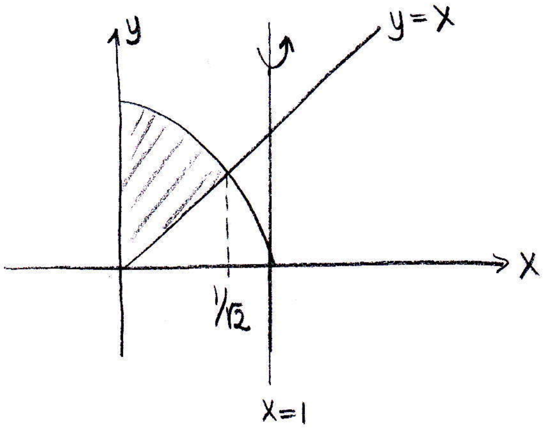
$x^2 = 1 \Rightarrow x = \mp 1$

$$\text{Area} = 2\left[\int_0^1 x^2\,dx + \int_1^{\sqrt{2}} (-x^2+2)\,dx\right]$$

$$= 2\left[\frac{x^3}{3}\Big|_0^1 + \left(-\frac{x^3}{3} + 2x\right)\Big|_1^{\sqrt{2}}\right]$$

$$= \frac{8(\sqrt{2}-1)}{3}$$

b)



Intersection point: $\sqrt{1-x^2} = x$

$1 - x^2 = x^2$

$x^2 = \frac{1}{2} \Rightarrow x = \frac{1}{\sqrt{2}}$

$$\text{Volume} = 2\pi \int \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dx$$

$$= 2\pi \int_0^{1/\sqrt{2}} (1-x)\left[\sqrt{1-x^2} - x\right] dx =$$

$$= 2\pi\left[\int_0^{1/\sqrt{2}} \sqrt{1-x^2}\,dx - \int_0^{1/\sqrt{2}} x\sqrt{1-x^2}\,dx - \int_0^{1/\sqrt{2}} (x - x^2)\,dx\right]$$

$x = \sin\theta$
$dx = \cos\theta\, d\theta$

$$= 2\pi\left[\int_0^{\pi/4} \cos^2\theta\, d\theta + \frac{1}{3}(1-x^2)^{3/2}\Big|_0^{1/\sqrt{2}} - \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^{1/\sqrt{2}}\right]$$

$$= 2\pi\left[\frac{\sin^{-1}x}{2} + \frac{x\sqrt{1-x^2}}{2}\Big|_0^{1/\sqrt{2}} + \frac{1}{3}\left[\left(\frac{1}{2}\right)^{3/2} - 1\right] - \left(\frac{1}{4} - \frac{1}{3.2^{3/2}}\right)\right)$$

$$= \frac{\pi}{}(-8 + 4\sqrt{2} + 3\pi)$$

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QUESTION 4 The blanks below will be filled by students. (Except the score)

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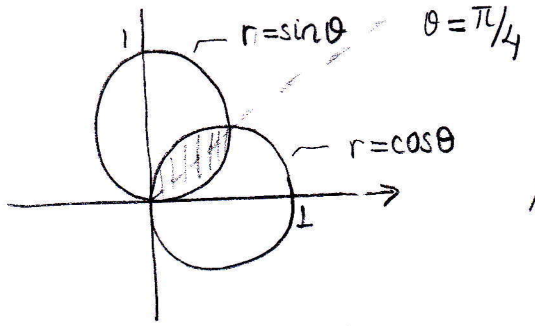
[10 pts] **a)** Let $r = \cos\theta$ and $r = \sin\theta$ be two curves given in polar coordinates.

Find the area of the region shared by the curves.

[10 pts] **b)** Investigate the convergence or divergence of the improper integral $\int_1^\infty \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx$

a)



$\sin\theta = \cos\theta \Rightarrow \theta = \pi/4$

$$\text{Area} = \frac{1}{2}\int_0^{\pi/4} (\sin\theta)^2\,d\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2} (\cos\theta)^2\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{1-\cos(2\theta)}{2}\,d\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2} \frac{1+\cos(2\theta)}{2}\,d\theta$$

$$= \frac{1}{4}\left(\theta - \frac{\sin(2\theta)}{2}\right)\Big|_0^{\pi/4} + \frac{1}{4}\left(\theta + \frac{\sin(2\theta)}{2}\right)\Big|_{\pi/4}^{\pi/2}$$

$$= \frac{\pi}{8} - \frac{1}{4}$$

b)

$$0 < \frac{1}{\sqrt{x}} < \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}} \qquad x \in [1,\infty)$$

$\int_1^\infty \frac{dx}{\sqrt{x}}$ is divergent since $p = \frac{1}{2} \leq 1$. By the Direct Comparison test $\int_1^\infty \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx$ is divergent.

OR Let $\frac{1}{x^{1/4}} = g(x)$, $\int_1^\infty \frac{dx}{x^{1/4}}$ $p = 1/4 < 1$ diverges

$$\lim_{x\to\infty} \frac{\frac{\sqrt{1+\sqrt{x}}}{x^{1/2}}}{\frac{1}{x^{1/4}}} = \lim_{x\to\infty} \frac{\sqrt{1+\sqrt{x}}}{x^{1/4}} = \lim_{x\to\infty} \frac{x^{1/4}\sqrt{\frac{1}{\sqrt{x}}+1}}{x^{1/4}} = 1$$

By limit Comparison T they both div or both conv

therefore $\int_1^\infty \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}$ diverges